THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2068 Honours Mathematical Analysis II
Tutorial 8

Date: 17 March, 2025

1. (Exercise 6.15 of [Rud76], see also Exercise 7.3 of [Ste08]) Suppose f is a real, continuously differentiable function on $[a,b], f(a) = f(b) = 0$, and

   C^1

   $$\int_a^b f^2(x)dx = 1.$$

   Prove that

   $$\int_a^b xf(x)f'(x)dx = -\frac{1}{2}, \quad \left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b x^2 f^2(x)dx\right) \geq \frac{1}{4}$$

   The inequality on the right is known as Heisenberg's inequality and is related to the uncertainty principle in physics.

2. (Exercise 7.3.8 of [BS11]) Let $0 < a < b$. Find $\int_a^b \lfloor x \rfloor dx$ where $\lfloor x \rfloor$ denotes the integer part of x for $x \geq 0$. Hint: For $x \geq 0$, consider $F(x) = (n-1)x - (n-1)n/2$ for $x \in [n-1, n), n \in \mathbb{N}$.

   $[a,b]$

3. (Exercise 7.3.10 of [BS11]) Let $f : [] \to \mathbb{R}$ be continuous on $[a,b]$ and let $\nu : [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$ with $\nu([c,d]) \subseteq [a,b]$. If we define $G(x) := \int_a^{\nu(x)} f$, show that $G'(x) = f(\nu(x))\nu'(x)$ for all $x \in [c,d]$.

4. (Exercise 7.3.14 of [BS11]) Show that there does not exist a continuously differentiable function f on $[0,2]$ such that $f(0) = -1, f(2) = 4, f'(x) \leq 2$ for $0 \leq x \leq 2$.

1. (Exercise 6.15 of [Rud76], see also Exercise 7.3 of [Ste08]) Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)dx = 1.$$

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The inequality on the right is known as Heisenberg's inequality and is related to the uncertainty principle in physics.

Pf: $\int_a^b xf(x)dx = \int_a^b u\,dv$ where $u = x$, $du = dx$, $dv = f(x)f'(x)\,dx \Rightarrow v = \frac{1}{2}f^2(x)$.

$$= uv\Big|_b^a - \int_a^b v\,du$$

$$= \overset{0}{\cancel{\frac{x}{2}f^2(x)}}\Big|_b^a - \int_a^b \frac{1}{2}f^2(x)\,dx \underset{\substack{\uparrow \\ \text{by } L^2 \text{ cond.}}}{=} -\frac{1}{2}$$

Schwarz's inequality with $xf(x)$ and $f'(x)$,

$$\left(\int_a^b x^2 f^2(x)\,dx\right)\left(\int_a^b (f'(x))^2\,dx\right) \geq \left(\int_a^b xf(x)f'(x)\,dx\right)^2 = \frac{1}{4} \;//.$$

2. (Exercise 7.3.8 of [BS11]) Let $0 < a < b$. Find $\int_a^b \lfloor x \rfloor dx$ where $\lfloor x \rfloor$ denotes the integer part of x for $x \geq 0$. Hint: For $x \geq 0$, consider $F(x) = (n-1)x - (n-1)n/2$ for $x \in [n-1, n), n \in \mathbb{N}$.

Pf: $F'(x) = n-1 = \lfloor x \rfloor$ for $x \in [n-1, n)$.

In order to apply FTC, we first need to check that F is continuous and differentiable except at finitely many points in $[a,b]$.

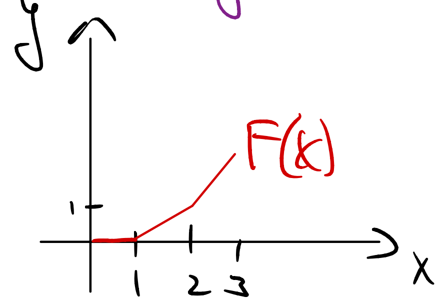
Clearly F is cts on $(n-1,n)$ for $n \in \mathbb{N}$.

(clearly, F does not have two sided derivatives at $n = 1, 2, 3, \ldots$ but is differentiable everywhere else)

At $x=0$, $n=1$ and

$$F(0) = (1-1)\cdot 0 - (1-1)\cdot\frac{1}{2} = 0.$$

$$\lim_{x \to 0^+} F(x) = (1-1)\cdot x - (1-1)\frac{1}{2} = 0.$$



At $x = n$, $F(n) = \frac{n(n-1)}{2}$

$$\lim_{x \to n^+} F(x) = (n-1)x - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$$

$$\lim_{x \to n^-} F(x) = (n-1)x - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}.$$ ✓.

So we apply FTC to obtain for $n-1 < a < n$, $m-1 < b < m$

$$\int_a^b \lfloor x \rfloor dx = F(b) - F(a) = (m-1)b - (m-1)\frac{m}{2} - (n-1)a + \frac{(n-1)n}{2}$$

$$= (m-1)\left(b - \frac{m}{2}\right) - (n-1)\left(a - \frac{n}{2}\right)$$

✓

3. (Exercise 7.3.10 of [BS11]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and let $\nu : [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$ with $\nu([c,d]) \subseteq [a,b]$. If we define $G(x) := \int_a^{\nu(x)} f$, show that $G'(x) = f(\nu(x))\nu'(x)$ for all $x \in [c,d]$.

Pf: Let $F(x) = \int_a^x f$. Then $G(x) = F(\nu(x))$

and $G'(x) = F'(\nu(x))\nu'(x)$ by chain rule and FTC.

4. (Exercise 7.3.14 of [BS11]) Show that there does not exist a continuously differentiable function f on $[0, 2]$ such that $f(0) = -1, f(2) = 4, f'(x) \leq 2$ for $0 \leq x \leq 2$.

Pf: By FTC, $f(x) = \int_0^x f(y)\,dy + f(0)$

$$\leq \int_0^x 2\,dy + f(0) = 2x - 1 \leq 3 \text{ for } x \in [0,2]$$

So $f(2) = 4$ is impossible.

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition*. Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.

[Rud76] Walter Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw-Hill Inc., 1976.

[Ste08] J. Michael Steele. *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. First Edition. Cambridge University Press, 2008.